Math 112: Introductory Real Analysis

§ Lecture 2 (Jan 29, 2025)

Try to give more examples!

Last time: ordered sets
upper bounds, bounded above
lower bounds, bounded below
least upper bound, greatest lower bound
(supremum) (infimum)
least-upper-bound property, greatest-lower-bound property
Thm: The l.u.b. property is equivalent to the g.l.b. property.

$\mathbb{Q}$ (and $\mathbb{R}$) are not just ordered sets, but have extra structures making them ordered fields.

Def A field is a set F with two binary operations, addition and multiplicatio[n] which satisfy the following "field axioms":

(A) Axioms for addition

- (commutativity) $x+y=y+x$ for all $x,y\in F$
- (associativity) $(x+y)+z = x+(y+z)$ for all $x,y,z\in F$
- (unit) There is $0\in F$ such that $0+x=x$ for every $x\in F$.
- (inverse) For every $x\in F$, there is $-x\in F$ such that $x+(-x)=0$.

(M) Axioms for multiplication

- (commutativity)
- (associativity)
- (unit) There is $1\in F$ such that $1\neq 0$ and $1\cdot x = x$ for every $x\in F$.
- (inverse) For every $x\in F$ with $x\neq 0$, there is $\frac{1}{x}\in F$ such that $x\cdot(\frac{1}{x})=1$.

(D) Distributive law

$$x\cdot(y+z) = xy + xz \text{ for all } x, y, z \in F.$$

E.g. $\mathbb{Q}$ is a field. ($\mathbb{R}$ is a field.)

$\mathbb{Z}$ is not.

Def An ordered field is a field F which is also an ordered set, such that

(i) $x+y < x+z$ for any $x, y, z \in F$ with $y<z$

(ii) $xy > 0$ for any $x, y \in F$ with $x>0$, $y>0$.

E.g. $\mathbb{Q}$ is an ordered field. (So is $\mathbb{R}$).

- The real field

Thm There exists an ordered field $\mathbb{R}$ which has the l.u.b. property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

proof sketch) There are several approaches, including Dedekind cuts and Cauchy sequences (by Cantor). Here we follow the approach using Dedekind cuts.

Def A real number is a non-empty proper subset $\alpha \subset \mathbb{Q}$ such that

(i) If $x \in \alpha$ and y is a rational number with $y<x$, then $y \in \alpha$.

(ii) There's no greatest element in α; that is, for any $x \in \alpha$, there is some $y \in \alpha$ with $y > x$.

3/

The set of real numbers is denoted by $\mathbb{R}$.

The idea is to model $\alpha \in \mathbb{R}$ by $\{x \in \mathbb{Q} \mid x < \alpha\} \subset \mathbb{Q}$.

E.g. $\sqrt{2} \in \mathbb{R}$ corresponds to $\{x \in \mathbb{Q} \mid x < \sqrt{2}\}$
$= \{x \in \mathbb{Q} \mid x < 0 \text{ or } x^2 < 2\}$.

Define an order on $\mathbb{R}$ by writing $\alpha < \beta$ whenever $\alpha \subsetneq \beta$.
(i.e. if β contains an upper bound of α)

(exercise: Show that $<$ is an order on $\mathbb{R}$.)

The ordered set $(\mathbb{R}, <)$ has the l.u.b. property.

(proof: If $A \subset \mathbb{R}$ is non-empty and bounded above,
let $\beta := \bigcup_{\alpha \in A} \alpha$.
Then β is a real number (exercise: show this))

We can also give it a structure of a field, making it into an ordered field.

$$\alpha + \beta := \{x + y_{\in \mathbb{Q}} \mid x \in \alpha, y \in \beta\},$$

$$0 := \{x \in \mathbb{Q} \mid x < 0\},$$

$$-\alpha := \{x \in \mathbb{Q} \mid -x \notin \alpha, -x \text{ is not the least element of } \mathbb{Q} \setminus \alpha\},$$

For $\alpha, \beta > 0$:

$$\alpha \cdot \beta := \{z \in \mathbb{Q} \mid z \leq 0 \text{ or } z = x \cdot y \text{ for some } x \in \alpha \text{ and } y \in \beta\},$$

$$1 := \{x \in \mathbb{Q} \mid x < 1\},$$

For $\alpha > 0$:

$$\frac{1}{\alpha} := \left\{x \in \mathbb{Q} \mid x \leq 0 \text{ or } x > 0 \text{ and } \tfrac{1}{x} \notin \alpha \text{ and } \tfrac{1}{x} \text{ is not the least element of } \mathbb{Q} \setminus \alpha\right\}$$ ■

Thm Any ordered field with the l.u.b. property is isomorphic to $\mathbb{R}$.

proof sketch) Suppose F is an ordered field with the l.u.b. property.

We need to construct an isomorphism $f: \mathbb{R} \to F$.

First, on $\mathbb{Z}$, define $f(0) = 0$,

$$f(n) = \underbrace{1 + \cdots + 1}_{n \text{ times}} \quad \text{for } n > 0,$$

$$f(-n) = -(\underbrace{1 + \cdots + 1}_{-n \text{ times}}) \quad \text{for } n < 0.$$

Then $f(m+n) = f(m) + f(n)$ and $f(m \cdot n) = f(m) \cdot f(n)$ for all $m, n \in \mathbb{Z}$.

On $\mathbb{Q}$, define $f\left(\frac{m}{n}\right) = \frac{f(m)}{f(n)}$. (Note, $n \neq 0 \Rightarrow f(n) \neq 0$)

Extend this to all $\mathbb{R}$ by defining

$$f(\alpha) = \sup \{ f(x) \in F \mid x \in \alpha \}.$$

(exercise: Show that f is well-defined; that is, when $\alpha \in \mathbb{Q}$ ($\alpha = \frac{m}{n}$), show that $\frac{f(m)}{f(n)} = \sup \{ f(r) \in F \mid r \in \mathbb{Q},\ r < \frac{m}{n} \}$.)

(exercise: Show that f is a field isomorphism and that it preserves order.)

$\blacksquare$

See Ch. 29 of Spivak's Calculus for the details.

- Some consequences of the least-upper-bound property of $\mathbb{R}$

<u>Thm</u> (Archimedean property of $\mathbb{R}$)

If $x, y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n such that

$$nx > y.$$

proof) If there were no such n, y would be an upper bound of

$$A := \{nx \mid n \in \mathbb{Z}_{>0}\} = \{x, 2x, 3x, \cdots\}$$

Then, $\sup A$ exists in $\mathbb{R}$.
Since $x > 0$, $\sup A - x$ is not an upper bound,
meaning $\sup A - x < mx$ for some $mx \in A$.
That would imply that $\sup A < (m+1)x \in A$,
which is a contradiction to $\sup A$ being an upper bound of A. ∎

<u>Thm</u> ($\mathbb{Q}$ is dense in $\mathbb{R}$)

For any $x, y \in \mathbb{R}$ with $x < y$, there is a $p \in \mathbb{Q}$ such that $x < p < y$.

proof) From the previous theorem, we know that there is $n \in \mathbb{Z}_{>0}$
such that $n(y-x) > 1$.
Again, from the Archimedean property, there are $m_1, m_2 \in \mathbb{Z}_{>0}$ such that

$$-m_2 < nx < m_1.$$

Hence there is an integer m with $-m_2 \leq m \leq m_1$ such that $m - 1 \leq nx < m$
Combining the inequalities, $nx < m \leq nx + 1 < ny$

$$\Rightarrow x < \frac{m}{n} < y \ \blacksquare$$

6/

Thm (existence of n-th roots)

For every real $x>0$ and every integer $n>0$, there is one and only one positive real y such that $y^n = x$.

proof) Uniqueness is clear, as $0<y_1<y_2$ implies $0<y_1^n<y_2^n$.

For the existence, let $E := \{t \in \mathbb{R} \mid t>0,\ t^n<x\}$.

Then E is non-empty (since $\frac{x}{1+x} \in E$)

and bounded above (since $1+x$ is an upper bound of E).

Therefore, $\sup E$ exists. Let's write $y := \sup E$ for simplicity.

To prove that $y^n = x$, we'll show that $y^n<x$ and $y^n>x$ lead to contradictions.

If $y^n<x$, we can choose h so that $0<h<\min\{1, \frac{x-y^n}{n(y+1)^{n-1}}\}$,

but then $(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x-y^n$

$\Rightarrow (y+h)^n < x \Rightarrow y+h \in E$ $\lightning$

If $y^n>x$, set $k = \frac{y^n-x}{ny^{n-1}}$. Then $0<k<y$, and

$y-k$ is an upper bound of E $\lightning$

(since if $t \geq y-k$, then $y^n - t^n \leq y^n-(y-k)^n < kny^{n-1} = y^n-x$

$\Rightarrow x<t^n \Rightarrow t \notin E$)

■